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# Thermalization of mutual information in hyperscaling violating backgrounds

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**ABSTRACT:** We study certain features of scaling behaviors of the mutual information during a process of thermalization, more precisely we extend the time scaling behavior of mutual information which has been discussed in [1] to time-dependent hyperscaling violating geometries. We use the holographic description of entanglement entropy for two disjoint system consisting of two parallel strips whose widths are much larger than the separation between them. We show that during the thermalization process, the dynamical exponent plays a crucial rule in reading the general time scaling behavior of mutual information (e.g., at the pre-local-equilibration regime). It is shown that the scaling violating parameter can be employed to define an effective dimension.

**KEYWORDS:** AdS-CFT Correspondence, Gauge-gravity correspondence

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## 1 Introduction

The AdS/CFT correspondence, sometimes called as gauge/gravity duality, is a conjectured relationship between quantum field theory and gravity. Precisely, in this correspondence, quantum physics of strongly correlated many-body systems is related to the classical dynamics of gravity which lives in one higher dimension. On the other hand according to

the AdS/CFT dictionary, an AdS geometry at the gravity side could only address the conformal symmetry of the dual field theory. However, the generalization of gauge/gravity correspondence to geometries which are not asymptotically AdS seems to be important as long as such extension may be related to the invariance under a certain scaling of dual field theory which does not even have conformal symmetry. Such generalization of AdS is actually motivated by consideration of gravity toy models in condensed matter physics (the application of such generalization can be found, for example, in [2]). A prototype of this generalization is a theory with the Lifshitz fixed point in which the spatial and time coordinates of a field theory have been scaled as

$$t \rightarrow \zeta^z t, \quad \vec{x} \rightarrow \zeta \vec{x}, \quad r \rightarrow \zeta r, \quad (1.1)$$

where  $z$  is the critical dynamical exponent. From the holographic duality point of view, for a  $(D+1)$ -dimensional theory, the corresponding  $(D+2)$ -dimensional gravity dual can be defined by the following metric

$$ds_{D+2}^2 = \frac{-dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{1}{r^2} \sum_{i=1}^D dx_i^2, \quad (1.2)$$

where in this paper the AdS radius is set to be one. Due to the anisotropy between space and time, it is clear that this metric can not be an ordinary solution of the Einstein equation, in fact one needs some sorts of matter fields to break the isotropy, e.g., by adding a massive vector field or a gauge field coupled to a scalar field [3–6]. In general by adding a dilaton with non trivial potential and an abelian gauge field to Einstein-Hilbert action (Einstein-Maxwell-Dilaton theory), one can find even more interesting metrics, in particular the following metric has been used frequently [7]

$$ds_{D+2}^2 = r^{\frac{2\theta}{D}} \left( \frac{-dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{1}{r^2} \sum_{i=1}^D dx_i^2 \right), \quad (1.3)$$

where  $\theta$  is hyperscaling violation exponent. This metric under the scale-transformation (1.1) transforms as  $ds \rightarrow \zeta^{\frac{\theta}{D}} ds$ . In a theory with hyperscaling violation, the thermodynamic parameters behave in such a way that they are stated in  $D - \theta$  dimensions; more precisely, in a  $(D+1)$ -dimensional theories which are dual to background (1.2), the entropy scales with temperature as  $T^{D/z}$ , however, in the presence of  $\theta$  namely dual to (1.3), it scales as  $T^{(D-\theta)/z}$  [7, 8]. Therefore one may associate an effective dimension to the theory and this becomes important in studying the log behavior of the entanglement entropy of system with Fermi surface in condense matter physics, explicitly it was shown that for  $\theta = D - 1$  for any  $z$ , the entanglement entropy exhibits a logarithmic violation of the area law [9]. On the other hand for such backgrounds time-dependency can be achieved by Vaidya metric with a hyperscaling violating factor. It is the main aim of this paper to investigate how entanglement (mutual information) spreads in time-dependent hyperscaling violating backgrounds.

Basically, the AdS-Vaidya metric is used to describe a gravitational collapse of a thin shell of matter in formation of the black hole. This metric in  $D + 2$  dimensions is given by

$$ds^2 = \frac{1}{\rho^2} \left( -f(\rho, v) dv^2 - 2d\rho dv + \sum_{i=1}^D dx_i^2 \right), \quad f(\rho, v) = 1 - m(v)\rho^{D+1}, \quad (1.4)$$

where  $\rho$  is the radial coordinate,  $x_i$ 's ( $i = 1, \dots, D$ ) are spatial boundary coordinates and, here, the mass  $m(v)$  is supposed to be an arbitrary function of the null coordinate<sup>1</sup>  $v$ . Holographic dual of such background can be described by a system which undergoes a sudden change which might be caused by turning on a uniform density of sources for a short time interval  $\delta t$  at  $t = 0$  and then turning it off. This process is called as a quantum quench which can excite the system to an excited state with non-zero energy density. Evolution towards an equilibrium state after a global quantum quench [10] is an example of the thermalization. During thermalization (being out of equilibrium) the usual thermodynamical quantities such as thermal entropy and pressure are not well-defined quantities. So that such an evolution cannot be studied within the context of standard thermodynamics, however, the entanglement entropy and the mutual information could potentially play a key role in probing the process of thermalization in those systems. Based on AdS/CFT correspondence, a global quench in the boundary theory can be described by a thin shell of matter starting from the boundary and collapsing to form a black hole and the corresponding metric is given by (1.4). On the other hand, the covariant holographic entanglement entropy proposal [11] suggests that entanglement entropy is proportional to the extremal surface in the bulk where this surface anchors on the boundary of entangling region on the conformal boundary of the bulk. This leads to the fact that entanglement entropy of the boundary region, will depend on time. In the process of thermalization, the local equilibrium which scaled by the horizon radius plays a role of defining a time scale when the system has ceased production of thermodynamic entropy though the entanglement entropy still increases [12].

In hyperscaling violating time-dependent geometries entanglement entropy has been studied in [13, 14]. In this work we study the time scaling behavior of the mutual information for a system after a global quantum quench. To do so we use Vaidya metric with hyperscaling violating factor, and the main aim of the present study is to extend the consideration of [1] in studying different scaling behaviors of the mutual information after a global quantum quench.

The organization of the paper is as follows. In the next section we will study the computations of holographic mutual information in a hyperscaling violating background. In section three we will study the scaling behaviors of the mutual information during the thermalization process. Last section is devoted to discussions. In the appendix we will review some related computations of the entanglement entropy.

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<sup>1</sup>We should mention that the mass function satisfies some boundary condition in time, e.g. for the initial time it is zero and for the late time it is a constant: transition from pure AdS to AdS black hole.

## 2 Mutual information in hyperscaling violating backgrounds

Entanglement entropy is a measure of storing quantum information in a quantum state. It is indeed a remarkable tool in studying quantum systems which has been deduced from the first principles of quantum mechanics. Beside the entanglement entropy, for two disjoint regions say as  $A$  and  $B$ , when we are interested in the amount of information that these two systems could share, the mutual information is mostly used. The mutual information can be expressed in terms of the entanglement entropy as

$$I(A, B) = S_A + S_B - S_{A \cup B}, \quad (2.1)$$

where  $S_A, S_B$  and  $S_{A \cup B}$  are entanglement entropies of the regions  $A, B$  and their union, respectively with the rest of the system. Entanglement entropy for two disjoint regions has been studied in [15–18],<sup>2</sup> therefore, one can use the results to study the mutual information. From the definition of mutual information and after making use of the subadditivity property of the entanglement entropy, it is evident that in computing the mutual information there is no UV divergency and it is always non-negative and it becomes zero if two system are uncorrelated. On the other hand from the AdS/CFT correspondence, one can show that there is a phase transition from zero to a positive value in the mutual information as one decreases the distance between two systems [20–22]. In ref. [1], for a certain entangling region namely for two strips, the various scaling behaviors of the mutual information during a process of thermalization after a global quantum quench have been studied. Here, in this paper we extend the results in parts to study the scaling behaviors of the mutual information in hyperscaling violating backgrounds. From the gauge/gravity duality the thermalization process after a global quantum quench is equivalent to the black hole formation at the gravity side due to a gravitational collapse. The corresponding metric is given by (1.4). For example entanglement entropy has been studied in [23–32] and mutual information in [33–36]. Now it is natural to study the time-dependency of the holographic entanglement entropy and mutual information in time-dependent hyperscaling violating background.

As mentioned, the hyperscaling violating geometries can be obtained by adding a scalar field with nontrivial potential (dilaton) and an abelian gauge field to the Einstein-Hilbert action. The general metric is given by

$$ds_{D+2}^2 = r^{-2\frac{\theta}{D}} \left( -r^{2z} f(r) dt^2 + \frac{dr^2}{r^{2f(r)}} + r^2 \sum_{i=1}^D dx_i^2 \right), \quad (2.2)$$

where  $z$  and  $\theta$  are the dynamical and hyperscaling violation exponents, respectively and

$$f(r) = 1 - \frac{m}{r^{D-\theta+z}}. \quad (2.3)$$

The above metric belongs to a neutral black brane solution with horizon radius  $r_H$ .<sup>3</sup> The important fact is that in the present model three free parameters appear: dimension of

<sup>2</sup>To study the holographic entanglement entropy of multiple strips in various holographic theories see also ref. [19].

<sup>3</sup> $f(r_H) = 0$  gives us the horizon radius.

the space-time  $D$ , scaling violating parameter  $\theta$  and dynamical exponent  $z$ . However, it is shown that the entanglement entropy and the mutual information up to an overall factor of  $L^{D-1}$  are sensitive to the dynamical exponent  $z$  and the effective dimension  $d$  which is defined by

$$d \equiv D - \theta + 1, \quad (2.4)$$

the overall factor can be fixed by a dimensional analysis. Moreover, in ref. [11], the authors have shown that the null energy condition at the gravity side imposes the following constraints on the free parameters

$$(D - \theta) \left( z - 1 - \frac{\theta}{D} \right) \geq 0, \quad (z - 1)(D + z - \theta) \geq 0, \quad (2.5)$$

in this paper we will consider  $z > 1$  and  $D > \theta$ .<sup>4</sup>

In what follows in this section, we will compute the holographic mutual information for two parallel infinite strips with the equal width  $\ell$  separated by a distance  $h$  in a  $D + 1$  dimensional field theory in both static and time-dependent hyperscaling violating backgrounds. We will also review the computation for entanglement entropy of a strip in such backgrounds in appendix A.

In ref. [20] it was shown that there would be a first order phase transition in computing the holographic mutual information when the distance between two strips increases. More precisely the value of  $\frac{h}{\ell}$  plays a crucial role in which above it the mutual information vanishes. This fact comes from the definition of entanglement entropy of the union  $A \cup B$  and for a given two strips with the widths  $\ell$  and distance  $h$ , from the holographic assumption point of view, this phase transition may be understood from the fact that, there are two minimal hypersurfaces associated with the entanglement entropy  $S(A \cup B)$ , consequently, the corresponding entanglement entropy behaves differently. This means that for two cases  $h \ll \ell$  and  $h \gg \ell$  one gets

$$S(A \cup B) = \begin{cases} S(2\ell + h) + S(h) & h \ll \ell, \\ 2S(\ell) & h \gg \ell, \end{cases} \quad (2.6)$$

where  $S(l)$  stands for the entanglement entropy of a strip with width  $l$ . Making use the above relation and also (2.1), clearly the mutual information becomes zero in the case of  $h \gg \ell$ , on the other hand for  $h \ll \ell$ , one finds

$$I(\ell, \ell, h) = 2S(\ell) - S(2\ell + h) - S(h) \equiv I. \quad (2.7)$$

In what follows we will also consider  $h \ll \ell$ . This indicates that in order to find the mutual information of two parallel strips, the entanglement entropy of three strips should be computed with widths  $h, \ell$  and  $2\ell + h$ . We emphasize that in this paper we consider the case in which the widths of two parallel strips are same, otherwise one has to compute four entanglement entropies corresponding to  $\ell_1, \ell_2, h$  and  $\ell_1 + \ell_2 + h$ .

First let us write the mutual information for two parallel strips in the vacuum state, it is indeed achieved from the entanglement entropy for a strip in a  $D + 1$ -dimensional CFT

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<sup>4</sup>We thank the referee for his/her comment on this point.

whose gravity dual is provided by the hyperscaling violating geometry. The corresponding entanglement entropies are given by (A.10), therefore one obtains

$$I_{\text{vac}} = \frac{L^{D-1}c_{d-1}}{4G_N} \left( -\frac{2}{\ell^{d-2}} + \frac{1}{(2\ell+h)^{d-2}} + \frac{1}{h^{d-2}} \right), \quad (2.8)$$

where  $d$  is the effective dimension and  $c_{d-1}$  is a coefficient are given by (2.4) and (A.11), respectively. The condition  $h \ll \ell$  guarantees the positivity of resultant mutual information. Note that as one increases the width of strips there is an upper limit for the mutual information given by

$$I_{\text{vac}}^{\text{max}} = \frac{L^{D-1}}{4G_N} \frac{c_{d-1}}{h^{d-2}}, \quad (2.9)$$

which is the absolute value of the finite part of the entanglement entropy for a strip with the width  $h$ .

On the other hand in order to compute the mutual information of the same strips in thermal states one can use the corresponding gravity dual which is provided by an AdS black brane hyperscaling violating metric (2.2). In general for the entanglement entropies there is no an analytic expression but in some certain limits, e.g.,  $\ell \ll \rho_H$  one can compute the entanglement entropy which is reviewed in appendix A. In this limit noting that since  $h \ll \ell$  one also has  $h \ll \rho_H$ , all the entanglement entropies involved in the computation of the mutual information, may be expanded as equation (A.8) and consequently the corresponding mutual information in this background is given by

$$I_{\text{BH}} = I_{\text{vac}} - \frac{L^{D-1}c_1^z}{16G_N(d-2)} \frac{(\ell+h)^{1+z}}{\rho_H^{d-1+z}}, \quad (2.10)$$

note that  $c_1$  is given by (A.9). From (2.10), one can say that for two static regions, mutual information is maximal when the system is in the vacuum state, namely  $I_{\text{vac}} > I_{\text{BH}}$ .<sup>5</sup>

Now let us consider the following case:  $h \ll \rho_H \ll \ell$ . One can say that the corresponding entanglement entropy for the region  $h$  should be approximated by equation (A.8), while for those of  $\ell$  and  $2\ell+h$  one has to use the large entangling region expansion given by equation (A.14), consequently the mutual information reads as

$$I = \frac{L^{D-1}}{4G_N} \left( \frac{c_{d-1}}{h^{d-2}} - \frac{c_2}{\rho_H^{d-2}} - \frac{h}{2\rho_H^{d-1}} - \frac{c_1^z}{4(d-2)\rho_H^{d-1+z}} h^{z+1} \right), \quad (2.11)$$

where  $c_2$  is a positive number which for some specific value of  $z$  and  $d$  it is calculated numerically in table (1). Note that in ref. [1], it was shown that although the term containing  $c_2$  is subleading in the expression of the entanglement entropy at large entangling region, but this term becomes significant in the expression of mutual information. Finally for  $\rho_H \ll h$  and  $\rho_H \ll \ell$  the mutual information is identically zero.

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<sup>5</sup>Note that in the process of thermalization after a quench this statement should be modified, mutual information undergoes a growing and then decreasing regimes, which will be discussed in the next section. We should mention that in this paper we only consider the excited states which are thermal, but when they are created by acting some local or non-local operators on the ground state, the answer is not so clear, for more details see [29].

### 3 Quantum quench: time evolution of mutual information in hyperscaling violating backgrounds

In this section we will study the scaling behavior of the holographic mutual information after a global quantum quench in a hyperscaling violating background. According to the proposal of covariant holographic entanglement entropy, time evolution of the entanglement entropy after a global quantum quench is controlled by the geometry around and inside the event horizon of the black hole. In ref. [12, 37] for large entangling region, it was shown that certain time intervals should be considered as pre-local-equilibration with quadratic time behavior, post-local-equilibration with linear growth, late-time equilibrium. To study the evolution of mutual information, for each part of the entanglement entropies of (2.1), one should take into account these time intervals as well. In what follows we are dealing with these intervals separately. To do so as in previous section, we shall consider two strips with widths  $\ell$  separated by distance  $h$  with condition  $h \ll \ell$ . We are interested in time evolution of the system, consequently, the covariant proposal of the holographic entanglement entropy of three strips with widths  $h, \ell$  and  $2\ell+h$  in the AdS-Vaidya metric (1.4) must be considered. This means that three hypersurfaces should be studied which each of them has crossing point and the turning point<sup>6</sup> as denoted by  $(\rho_{i\,c}, \rho_{i\,t})$  with  $i = 1, 2, 3$ . If one uses the entanglement entropy as a probe, the radius of the horizon  $\rho_H$ , and the size of the entangling region are used as scales to address the time evolution of the system. However in our case beside the radius of the horizon,  $\frac{\ell}{2}$ ,  $\frac{h}{2}$  and  $\ell + \frac{h}{2}$  play the crucial rule. So that depending on the size of  $\rho_H$  is larger or smaller than the entangling regions and noting that  $h \ll \ell$ , one may distinguish four main possibilities for the order of scales as follows

- $\rho_H \ll \frac{h}{2}$ ,
- $\frac{h}{2} \ll \rho_H \ll \frac{\ell}{2} < \ell + \frac{h}{2}$ ,
- $\frac{h}{2} \ll \frac{\ell}{2} < \rho_H < \ell + \frac{h}{2}$ ,
- $\frac{h}{2} \ll \frac{\ell}{2} < \ell + \frac{h}{2} \ll \rho_H$ .

In what follows we will consider the cases separately.

#### 3.1 First regime: $\rho_H \ll \frac{h}{2}$

In order to study the time evolution of mutual information we should compute the co-dimension two hypersurfaces of the all entangling regions which are associated with the entanglement entropies appeared in equation (2.7). But in this case it is important to note that the hypersurfaces cross the null shell and could probe the  $v < 0$  region. In this case five separated time intervals might be considered as:

$$t \ll \rho_H^z, \quad \rho_H^z \ll t \ll \rho_H^{z-1} \frac{h}{2}, \quad \rho_H^{z-1} \frac{h}{2} \ll t \ll \rho_H^{z-1} \frac{\ell}{2},$$

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<sup>6</sup>The crossing point refers to a point where the hypersurface intersects the null shell whereas  $\rho_{i\,t}$  stands for the turning point of the extremal hypersurface in the bulk.



$$\rho_H^{z-1} \frac{\ell}{2} < t < \rho_H^{z-1} \left( \ell + \frac{h}{2} \right), \quad \text{Saturation time.} \quad (3.1)$$

Let us study the time evolution in each case.

### 3.1.1 $t \ll \rho_H^z$

At the very early time after the quench, all the co-dimension two hypersurfaces in the bulk cross null shell where the crossing points are very close to the boundary this means  $\frac{\rho_c}{\rho_H} \ll 1$ , so that  $\rho_{i\,c}$  approximately are the same and  $\rho_{i\,c} \approx \rho_c$ . Therefore one can expand  $\ell$  and  $t$  which are given by (A.34) and also  $\mathcal{A}$  in this limit and hence one obtains

$$t \approx \rho_t^z \int_0^{\frac{\rho_c}{\rho_t}} d\xi \frac{\xi^{z-1}}{f(\rho_t \xi)} = \frac{\rho_c^z}{z} \left( 1 + \frac{z}{d-1+2z} \left( \frac{\rho_c}{\rho_H} \right)^{d-1+z} + \mathcal{O} \left( \left( \frac{\rho_c}{\rho_H} \right)^{2(d-1+z)} \right) \right), \quad (3.2)$$

where  $\xi \equiv \frac{\rho}{\rho_t}$ . This leads to the following expression for the area

$$\mathcal{A} \approx L^{D-1} \left[ \frac{1}{(d-2)\epsilon^{d-2}} - \frac{c_{d-1}}{\ell^{d-2}} + \frac{1}{2(z+1)} \frac{\rho_c^{z+1}}{\rho_H^{d+z-1}} \left( 1 + \mathcal{O} \left( \left( \frac{\rho_c}{\rho_H} \right)^{d+z-1} \right) \right) \right], \quad (3.3)$$

where  $\epsilon$  is the UV cutoff and  $c_{d-1}$  is given by (A.11). Making use of (3.2) and (3.3) and noting that  $m = \rho_H^{1-d-z}$ , at the leading order one finds

$$S(\ell_i) \approx \frac{L^{D-1}}{4G_N} \left[ \frac{1}{(d-2)\epsilon^{d-2}} - \frac{c_{d-1}}{\ell_i^{d-2}} + \frac{m}{2(z+1)} (zt)^{1+\frac{1}{z}} \right]. \quad (3.4)$$

As one sees this expression is independent of the hyperscaling violating factor  $\theta$ . Note that through out this section we use a notion in which  $i = 1, 2, 3$  where  $l_1 = h$ ,  $l_2 = \ell$  and  $l_3 = 2\ell + h$ . Plugging these expressions into equation (2.7), one finds

$$I = \frac{L^{D-1} c_{d-1}}{4G_N} \left( -\frac{2}{\ell^{d-2}} + \frac{1}{h^{d-2}} + \frac{1}{(2\ell + h)^{d-2}} \right) + \mathcal{O}(t^{2d}) = I_{\text{vac}} + \mathcal{O}(t^{2d}). \quad (3.5)$$

One observes that the mutual information starts from its value in the vacuum,  $I_{\text{vac}}$ , and remains fixed up to order of  $\mathcal{O}(t^{2d})$  at the early times.

### 3.1.2 Steady behavior: $\rho_H^z \ll t \ll \rho_H^{z-1} \frac{h}{2}$

In this time interval, in the case of computing the entanglement entropy a significant observation of linear growth with time has been observed in [12, 37], and the extension to the hyperscaling violating backgrounds was done in [13]. In ref. [1] it was argued that the mutual information inherits this linear behavior as well. Actually there is a critical extremal surface which is responsible for the linear growth in this time interval. More precisely, equation (A.30) might be thought of as the energy conservation law for a one-dimensional dynamical system whose effective action is given by  $V_{\text{eff}}(\rho)$  with dynamical variable  $\rho$ . Now, for  $\rho_t$  being a fixed turning point,  $\rho_c$  can play as a free parameter which may be tuned to a particular value of  $\rho_c = \rho_c^*$  in a way that

$$\left. \frac{\partial V_{\text{eff}}(\rho)}{\partial \rho} \right|_{\rho_m, \rho_c^*} = 0, \quad V_{\text{eff}}(\rho)|_{\rho_m, \rho_c^*} = 0, \quad (3.6)$$

where  $\rho_m$  is a point which minimizes the effective action. This indicates that if the hypersurface intersects the null shell at the critical point, it remains fixed at  $\rho_m$ . One can show that in this time interval the main contributions to  $\ell$ ,  $t$  and  $\mathcal{A}$  come from a hypersurface which is closed to the critical extremal hypersurface. Now we want to compute the  $\ell$ ,  $t$  and  $\mathcal{A}$  around the critical extremal hypersurface, to do so let us consider  $\rho_c = \rho_c^*(1 - \delta)$  in which  $\delta \ll 1$  when  $\rho \rightarrow \rho_m$ . Assuming that both  $\frac{\rho_c^*}{\rho_t}$  and  $\frac{\rho_m}{\rho_t}$  are very smaller than one, the expression of (A.34) may be approximated in the hyperscaling violating background as [13]

$$t \approx -\rho_t^z \frac{\xi_m^{2z-2} E^*}{f(\rho_t \xi_m) \sqrt{\frac{1}{2} V_{\text{eff}}''(\rho_t \xi_m)}} \ln \delta, \quad \frac{\ell}{2} \approx c \rho_t - \frac{\rho_t}{\sqrt{\frac{1}{2} V_{\text{eff}}''(\rho_t \xi_m)}} \ln \delta, \quad (3.7)$$

where  $V_{\text{eff}}''(\rho_t \xi_m) = \frac{\partial^2 V_{\text{eff}}(\rho_t \xi_m)}{\partial \xi^2} \big|_{\xi_m, \xi_c^*}$  and  $c = \sqrt{\pi} \frac{\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})}$ , and we have defined  $E^*$  as

$$E^* = -\left(\frac{\rho_t}{\rho_m}\right)^{z-1} \sqrt{-f(\xi_m) \left( \left(\frac{\rho_t}{\rho_m}\right)^{2z-2} - 1 \right)}. \quad (3.8)$$

On the other hand from (A.35) one finds

$$\frac{\mathcal{A}_{d-1}}{L^{D-1}} \approx \frac{1}{(d-2)\epsilon^{d-2}} - \frac{c_{d-1}}{\ell_i^{d-2}} + \frac{1}{\rho_t^{d+z-2}} \frac{f(\rho_t \xi_m)}{\xi_m^{2(d+z-2)} E^*} t, \quad (3.9)$$

$$= \frac{\mathcal{A}_{\text{vac}}}{L^{D-1}} + \frac{1}{\rho_t^{d+z-2}} \frac{f(\rho_t \xi_m)}{\xi_m^{2(d+z-2)} E^*} t. \quad (3.10)$$

Plugging these relations in (3.8), for large  $\rho_t$  the entanglement entropy reads

$$S \approx \frac{L^{D-1}}{4G_N} \left[ \frac{1}{(d-2)\epsilon^{d-2}} - \frac{c_{d-1}}{\ell_i^{d-2}} + \frac{\sqrt{-f(\rho_m)}}{\rho_m^{d+z-2}} t + \dots \right]. \quad (3.11)$$

Note that by making use of equation (3.6), one can obtain  $\rho_m$  and  $\rho_c^*$  in terms of the radius of horizon.

The mutual information is then obtained from equation (2.7) as follows

$$I = I_{\text{vac}} + \frac{L^{D-1}}{4G_N} \left( 2 \frac{\sqrt{-f(\rho_{2m})}}{\rho_{2m}^{d+z-2}} - \frac{\sqrt{-f(\rho_{1m})}}{\rho_{1m}^{d+z-2}} - \frac{\sqrt{-f(\rho_{3m})}}{\rho_{3m}^{d+z-2}} \right) t + \dots. \quad (3.12)$$

The important fact is that since we are dealing with the large entangling regions, the turning points of all hypersurfaces are large, one can deduce that  $\rho_{im} = \rho_m$ . As a result, the second term in the above equation vanishes leading to a constant mutual information in this time interval too. Thus starting from a static solution one gets almost constant mutual information all the way from  $t = 0$  to  $t \sim \rho_H^{z-1} \frac{h}{2}$ .

### 3.1.3 Linear growth: $\rho_H^{z-1} \frac{h}{2} \ll t \ll \rho_H^{z-1} \frac{\ell}{2}$

In this time interval the corresponding entanglement entropy of  $h$  saturates. The saturation time can indeed be obtained by making use of the expression of the saturated entanglement

entropy and the intermediate entanglement entropy (3.11), which one obtains the saturation time as

$$t_s \sim \frac{h}{2} \rho_H^{z-1} - c_2 \rho_H^z + c_{d-1} \frac{\rho_H^{d+z-2}}{h^{d-2}}, \quad (3.13)$$

where  $c_{d-1}$  is given by (A.11). So that one should use the saturated entanglement entropy associated with  $h$  which is discussed in appendix and given by

$$S(h) \approx \frac{L^{D-1}}{4G_N} \left( \frac{1}{(d-2)\epsilon^{d-2}} + \frac{h}{2\rho_H^{d-1}} - \frac{c_2}{\rho_H^{d-2}} \right). \quad (3.14)$$

Whereas the entanglement entropies associated with the entangling regions  $\ell$  and  $2\ell + h$  are still increasing linearly with time which are given by equation (3.11). Gathering all the results into equation (2.7) one finds

$$I = \frac{L^{D-1}}{4G_N} \left[ -\frac{2c_{d-2}}{\ell^{d-2}} + \frac{c_{d-2}}{(2\ell+h)^{d-2}} - \frac{1}{\rho_H^{d-1}} \frac{h}{2} + \frac{c_2}{\rho_H^{d-2}} + \left( 2 \frac{\sqrt{-f(\rho_{2m})}}{\rho_{2m}^{d+z-2}} - \frac{\sqrt{-f(\rho_{3m})}}{\rho_{3m}^{d+z-2}} \right) t + \dots \right], \quad (3.15)$$

which can be recast into the following form

$$I = I_{\text{vac}} + \frac{L^{D-1}}{4G_N} \left( \frac{c_2}{\rho_H^{d-2}} - \frac{c_{d-2}}{h^{d-2}} \right) + \frac{L^{D-1}}{4G_N \rho_H^{d-1}} \left( v_E t - \frac{h}{2} \right), \quad (3.16)$$

where

$$v_E \equiv \rho_H^{d-1} \left( 2 \frac{\sqrt{-f(\rho_{2m})}}{\rho_{2m}^{d+z-2}} - \frac{\sqrt{-f(\rho_{3m})}}{\rho_{3m}^{d+z-2}} \right). \quad (3.17)$$

It is worth to mention that in hyperscaling violating backgrounds in general for  $z \neq 1$  the turning point in the bulk can not be fixed only by  $\rho_m$ . Namely for a fixed  $\rho_H$ , the radial coordinate which minimizes the effective potential is a function of both  $\rho_t$  and  $\rho_c$  [13]. In other words the effective potential becomes minimum at  $\rho_m$  which one finds

$$\rho_t^{2(d-1)} = \rho_m^{2(d-1)} \frac{2\rho_m f'(\rho_m) + (z-1) \left( \frac{\rho_c}{\rho_H} \right)^{2d-2+2z} \left( \frac{\rho_m}{\rho_c} \right)^{2(z-1)}}{2\rho_m f'(\rho_m) - 4(d-1)f(\rho_m) + (z-1) \left( \frac{\rho_c}{\rho_H} \right)^{2d-2+2z} \left( \frac{\rho_m}{\rho_c} \right)^{2(d-2+z)}}. \quad (3.18)$$

However, for the critical extremal hypersurface which is defined by the conditions (3.6), imposing the condition of critical point where at the minimum point the effective potential is also zero, results in

$$\rho_t^{2(d-1)} = \rho_m^{2(d-1)} \frac{4f(\rho_m) + \left( \frac{\rho_c^*}{\rho_H} \right)^2 \left( \frac{\rho_m}{\rho_c} \right)^{2(z-1)}}{4f(\rho_m) + \left( \frac{\rho_c^*}{\rho_H} \right)^2 \left( \frac{\rho_m}{\rho_c} \right)^{2(d-2+z)}}, \quad (3.19)$$

this relation can indeed fix  $\rho_m$  and  $\rho_c^*$ , and by solving relations (3.18) and (3.19) one can find  $\rho_c^*$  and  $\rho_m$ . For large entangling region (or at large  $\rho_t$  limit) assuming that both  $\rho_m$  and  $\rho_c^*$  remain finite one gets

$$\frac{\rho_m}{\rho_H} = \left( \frac{2(d+z-2)}{d+z-3} \right)^{\frac{1}{d+z-1}}, \quad \frac{\rho_c^*}{\rho_H} = 2\sqrt{\frac{d+z-1}{d+z-3}} \left( \frac{d+z-3}{2(d+z-2)} \right)^{\frac{d+z-2}{d+z-1}}. \quad (3.20)$$

Since we are dealing with large entangling regions one can say that  $\rho_{im} = \rho_m$ , then after making use of the above relations one can show that

$$v_E = \sqrt{\frac{d+z-1}{d+z-3}} \left( \frac{d+z-3}{2(d+z-2)} \right)^{\frac{d+z-2}{d+z-1}} \rho_H^{1-z}. \quad (3.21)$$

Note that the mutual information (2.1) is positive as long as  $\rho_H \ll \frac{h}{2}$  and  $\rho^{z-1} \frac{h}{2} \ll t$ , moreover in this time interval the mutual information for  $d+z > 3$  is always bigger than  $I_{\text{vac}}$ . It is clear that the mutual information grows linearly with time. It is also worth mentioning that the existence of subleading term  $\frac{c_2}{\rho_H^{d-2}}$  in obtaining the entanglement entropy in this time interval play an important role in getting a positive mutual information.

In this time interval during the thermalization of system the mutual information grows linearly with time and there is an upper limit of the mutual information which takes place when the entanglement entropy associated with the entangling region  $\ell$  saturates to its equilibrium value at  $t \sim \frac{\ell}{2} \rho_H^{z-1} - c_2 \rho_H^z + c_{d-1} \frac{\rho_H^{d+z-2}}{\ell^{d-2}}$ . More precisely setting

$$v_E t_{\text{max}}^{(1)} \sim \frac{\ell}{2} - c_2 \rho_H + c_{d-1} \frac{\rho_H^{d-1}}{\ell^{d-2}}, \quad (3.22)$$

one finds

$$I_{\text{max}}^{(1)} \approx I_{\text{vac}} + \frac{L^{D-1}}{4G_N} \left( \frac{c_{d-1}}{\ell^{d-2}} - \frac{c_{d-1}}{h^{d-2}} \right) + \frac{L^{D-1}}{4G_N \rho_H^{d-1}} \left( \frac{\ell}{2} - \frac{h}{2} \right). \quad (3.23)$$

Here  $t_{\text{max}}^{(1)}$  is the time when the mutual information reaches its maximum value  $I_{\text{max}}^{(1)}$ . Note that this maximum value is independent of  $z$ .

### 3.1.4 Linear decreasing: $\rho_H^{z-1} \frac{\ell}{2} < t < \rho_H^{z-1} (\ell + \frac{h}{2})$

In this time interval, one deduces that both entanglement entropies  $S(\ell)$  and  $S(h)$  should be approximated by their saturated values (noting that the saturation of  $S(\ell)$  takes place at  $\frac{\ell}{2} - c_2 \rho_H + c_{d-1} \frac{\rho_H^{d-1}}{\ell^{d-2}}$  and in this time  $S(h)$  is already saturated). On the other hand corresponding entanglement entropy of  $2\ell + h$  still grows linearly with time which is given by (3.11). Plugging these three entanglement entropies in (2.7) one finds

$$I = \frac{L^{D-1}}{4G_N} \left( \frac{c_{d-1}}{(2\ell + h)^{d-2}} - \frac{c_2}{\rho_H^{d-2}} + \frac{\ell}{\rho_H^{d-1}} - \frac{h}{2\rho_H^{d-1}} - \frac{\sqrt{-f(\rho_{3m})}}{\rho_{3m}^{d+z-2}} t \right), \quad (3.24)$$

which may be simplified as follows

$$I \approx I_{\text{max}}^{(1)} + \frac{L^{D-1}}{4G_N} \left( \frac{c_{d-1}}{\ell^{d-2}} - \frac{c_2}{\rho_H^{d-2}} \right) + \frac{L^{D-1}}{4G_N \rho_H^{d-1}} \left( \frac{\ell}{2} - v_E t \right). \quad (3.25)$$

The mutual information in this time interval declines linearly with time and is positive for  $t < \rho_H^{z-1} (\ell + \frac{h}{2})$  and also  $I < I_{\text{max}}^{(1)}$ .

### 3.1.5 Saturation

Final step in thermalization takes place if time passes enough when all the entanglement entropies  $S(\ell_i)$  saturate to their equilibrium value. In fact the last one is  $S(2\ell + h)$  with the saturation time as given by

$$t \sim \left( \ell + \frac{h}{2} \right) \rho_H^{z-1} - c_2 \rho_H^z + c_{d-1} \frac{\rho_H^{d+z-2}}{(2\ell + h)^{d-2}}. \quad (3.26)$$

Accordingly the mutual information will also saturates to its equilibrium value. However in ref. [1] it was shown that the time when the mutual information reaches to its equilibrium value is not just the same as the saturation time of  $S(2\ell + h)$ . But thanks to the proposed conditions in which both the width of strips  $\ell$  and distance between them  $h$  are large compared to the radius of the horizon namely,  $\rho_H \ll \ell$  and  $\rho_H \ll h$ , the mutual information becomes zero at the end of the thermalization process. Indeed assuming the mutual information decreases all the way till it becomes zero, from equation (3.24), one should set

$$I_{\text{sat}}^{(1)} \approx \frac{L^{D-1}}{4G_N} \left( \frac{c_{d-1}}{(2\ell + h)^{d-2}} - \frac{c_2}{\rho_H^{d-2}} + \frac{\ell}{\rho_H^{d-1}} - \frac{h}{2\rho_H^{d-1}} - \frac{v_E}{\rho_H^{d-1}} t_s^{(1)} \right) = 0, \quad (3.27)$$

so that the saturation time reads

$$v_E t_s^{(1)} \approx \ell - \frac{h}{2} - c_2 \rho_H + \frac{c_{d-1} \rho_H^{d-1}}{(2\ell + h)^{d-2}}, \quad (3.28)$$

or for large entangling regions one write  $t_s^{(1)} \approx (\ell - \frac{h}{2}) \rho_H^{z-1} - c_2 \rho_H^z$ , comparing with the saturation time of the  $S(2\ell + h)$  reveals that the mutual information saturates earlier than the saturation time of the entanglement entropy of a strip with width  $2\ell + h$ .<sup>7</sup>

To summarize the results of first regime in which  $t \ll \rho_H^z$ , one sees that the mutual information in the hyperscaling violating backgrounds after a quantum quench undergoes four main situations: it starts from its value in the vacuum and remains almost constant up to  $t \sim \rho_H^{z-1} \frac{h}{2}$ , after that it grows with time linearly till its maximum value which takes place at  $t_{\text{max}}^{(1)}$ . Then the mutual information decreases linearly with time till it becomes zero at the saturation time which takes place approximately at  $t_s^{(1)} \sim (\ell - \frac{h}{2}) \rho_H^{z-1} - c_2 \rho_H^z$ .

### 3.2 Second regime: $\frac{h}{2} \ll \rho_H \ll \frac{\ell}{2} < \ell + \frac{h}{2}$

In this regime, to study the behavior of the mutual information, similar to the previous regime, five time intervals can be considered separately, as stated below

$$\begin{aligned} t &\ll \rho_H^{z-1} \frac{h}{2}, & \frac{h}{2} \rho_H^{z-1} &\ll t \ll \rho_H^z, & \rho_H^z &\ll t \ll \rho_H^{z-1} \frac{\ell}{2}, \\ \rho_H^{z-1} \frac{\ell}{2} &< t < \rho_H^{z-1} \left( \ell + \frac{h}{2} \right), & & & \text{Saturation time.} \end{aligned} \quad (3.29)$$

At the very early time  $t \ll \rho_H^{z-1} \frac{h}{2}$ , the behavior of  $S(\ell_i)$  is similar to what discussed in previous case, so that the behavior of the entanglement entropy at the very early time is given by (3.12), which means that it remains constant till  $t \ll \rho_H^{z-1} \frac{h}{2}$ .

<sup>7</sup>In Vaidya geometry, similar result has been observed numerically in [33–36].

### 3.2.1 Non-linear growth: $\frac{h}{2}\rho_H^{z-1} \ll t \ll \rho_H^z$

Having noted that  $\frac{h}{2} \ll \rho_H$ , the corresponding co-dimension two hypersurface of region  $h$  cannot probe the  $v < 0$  region, namely the corresponding hypersurface remains always at  $v > 0$  region which is, indeed, a static hyperscaling violating AdS black brane. Thus for  $S(h)$  one should use its equilibrium value which is given by equation (A.8). However, due to the  $t \ll \rho_H^{z-1}\frac{\ell}{2}$  situation, one can say that  $S(\ell)$  and  $S(2\ell + h)$  are still at the early times which has been given by equation (3.4). Plugging these three entropies in (2.7) one gets

$$I \approx I_{\text{vac}} + \frac{L^{D-1}}{4G_N\rho_H^{d-1+z}} \left( -c_1^z h^{z+1} + \frac{(zt)^{1+1/z}}{2z+2} \right), \quad (3.30)$$

showing that the mutual information has a non-linear growth up to  $t \sim \rho_H^z$ .

It is worth mentioning that even though we are dealing with hyperscaling violating geometry, among three free parameters of the theory, namely  $D$ ,  $\theta$  and  $z$ , non-linear evolution of mutual information in this time interval is independent of hyperscaling violating parameter  $\theta$ . In fact,  $\theta$  leads to an effective dimension as  $D - \theta$  and nontrivial behaviors of the resultant mutual information are the same as that of Lifshitz ( $\theta = 0$ ,  $z \neq 0$ ) geometry in  $D + 1$  dimensions. On the other hand it was argued that time scaling behavior of the entanglement entropy is dimension-independent [12], in this sense one can say that the non-linear behavior of mutual information (scaling as  $t^{1+1/z}$ ) is independent of  $\theta$  and one can say  $z$  appears in thermodynamical quantities like those in Lifshitz backgrounds.

### 3.2.2 Linear growth $\rho_H^z \ll t \ll \rho_H^{z-1}\frac{\ell}{2}$

For  $S(h)$ , there is no change and hence it is still given by equation (A.8). On the other hand the system has reached a local equilibrium and due to  $\rho_H \ll \frac{\ell}{2}$ , the entanglement entropies of  $S(\ell)$  and  $S(2\ell + h)$  should be approximated by equation (3.11). Thanks to the fact that the entangling regions are large so that  $\rho_{im} = \rho_m$ , one obtains the following expression for the mutual information

$$I \approx I_{\text{vac}} + \frac{L^{D-1}}{4G_N\rho_H^{d-1}} \left( v_E t - \frac{c_1^z}{4(d-2)\rho_H^z} h^{z+1} \right). \quad (3.31)$$

In this time interval, the conditions  $h \ll \rho_H$  and  $\rho_H^z \ll t$  guarantee that the resultant mutual information will be positive and bigger than  $I_{\text{vac}}$ . The mutual information linearly grows with time till when the  $S(\ell)$  saturates to its equilibrium value. Actually this happens at

$$t_{\text{max}}^{(2)} \approx \frac{\ell}{2}\rho_H^{z-1} - c_2\rho_H^z + c_{d-1}\frac{\rho_H^{d+z-2}}{\ell^{d-2}}. \quad (3.32)$$

One can use equation (3.32) to estimate the maximum value of the mutual information as follows

$$I_{\text{max}}^{(2)} \approx I_{\text{vac}} + \frac{L^{D-1}}{4G_N\rho_H^{d-1}} \left( \frac{\ell}{2} - c_2\rho_H + c_{d-1}\frac{\rho_H^{d-1}}{\ell^{d-2}} - \frac{c_1^z}{4(d-2)\rho_H^z} h^{z+1} \right). \quad (3.33)$$

### 3.2.3 Linear decreasing: $\rho_H^{z-1} \frac{\ell}{2} < t < \rho_H^{z-1} (\ell + \frac{h}{2})$

The main change in the present time interval comes from the fact that entanglement entropy  $S(\ell)$  is saturated to its equilibrium value. However, due to the size of entangling region  $\ell$ , one should follow the situation which has been already discussed in appendix to estimate the corresponding entanglement entropy which is given by

$$S(\ell) \approx \frac{L^{D-1}}{4G_N} \left( \frac{1}{(d-2)\epsilon^{d-2}} + \frac{\ell}{2\rho_H^{d-1}} - \frac{c_2}{\rho_H^{d-2}} \right). \quad (3.34)$$

Noting that there is no change in reading the  $S(2\ell+h)$  and  $S(h)$  which are given by (3.11) and (A.8) respectively. Therefore, the mutual information has a decreasing regime in this time interval as stated below

$$I \approx I_{\max}^{(2)} + \frac{L^{D-1}}{4G_N} \left( \frac{c_{d-1}}{\ell^{d-2}} - \frac{c_2}{\rho_H^{d-2}} \right) + \frac{L^{D-1}}{4G_N \rho_H^{d-1}} \left( \frac{\ell}{2} - v_E t \right). \quad (3.35)$$

Note that the mutual information is positive and also  $I < I_{\max}^{(2)}$ .

### 3.2.4 Saturation

After a long time the system which we are dealing with reaches to thermal state whose gravity dual is provided by a hyperscaling violating AdS black brane. However, as previous case having noted the condition  $\frac{h}{2} \ll \rho_H \ll \frac{\ell}{2}$ , one can use (2.11) for the mutual information and hence the equilibrium value of the mutual information becomes

$$I_{\text{sat}}^{(2)} \approx \frac{L^{D-1}}{4G_N} \left( \frac{c_{d-1}}{h^{d-2}} - \frac{c_2}{\rho_H^{d-2}} - \frac{h}{2\rho_H^{d-1}} - \frac{c_1^z}{4(d-2)\rho_H^{d-1+z}} h^{z+1} \right). \quad (3.36)$$

The above expression for mutual information can be written as

$$I_{\text{sat}}^{(2)} = I_{\text{vac}} - \frac{L^{D-1}}{4G_N} \left( \frac{c_2}{\rho_H^{d-2}} + \frac{h}{2\rho_H^{d-1}} + \frac{c_1^z}{4(d-2)\rho_H^{d-1+z}} h^{z+1} + \frac{c_{d-1}}{(2\ell+h)^{d-2}} - \frac{2c_{d-1}}{\ell^{d-2}} \right) \quad (3.37)$$

which shows that in this case with the condition  $\frac{h}{2} \ll \rho_H \ll \frac{\ell}{2}$  the expression in the parentheses is always positive leading to the fact that  $I_{\text{sat}}^{(2)} < I_{\text{vac}}$ .

In order to estimate the time in which the mutual information saturates in this regime, we should just follow the recipe in which the mutual information decreases linearly with time till it reaches its equilibrium value. In other words the saturating time can be obtained by equating equations (3.35) and (3.36), and making use of (3.21), one finds the saturation time as

$$t_s^{(2)} \approx \left( \ell + \frac{h}{2} \right) \rho_H^{z-1} - c_2 \rho_H^z. \quad (3.38)$$

Let us summarize the results of present regime  $\frac{h}{2} \ll \rho_H \ll \frac{\ell}{2} < \ell + \frac{h}{2}$ . The time evolution of the mutual information undergoes five main phases: it starts from the value in vacuum and remains almost constant up to  $t \sim \rho_H^{z-1} \frac{h}{2}$ , then it grows non-linearly with time as  $t^{1+1/z}$  till  $t \sim \rho_H^z$ . After that it linearly grows till its maximum value which takes place at  $t_{\max}^{(2)}$ . After the maximum value, it decreases linearly with time and finally it saturates to a constant value at the saturation time  $t_s^{(2)}$ .

### 3.3 Third regime: $\frac{h}{2} \ll \frac{\ell}{2} < \rho_H < \ell + \frac{h}{2}$

In this regime the related co-dimension two hypersurfaces of  $S(h)$  and  $S(\ell)$  cannot probe the region near and behind the horizon and entangling regions saturate to their equilibrium values before the system reaches a local equilibrium. Thus only  $S(2\ell + h)$  grows linearly with time before it reaches its equilibrium value.

The very early time behavior is in fact the same as the similar case of the previous regime, so that one can say that the mutual information has a fixed value in the vacuum till  $t \sim \rho_H^{z-1} \frac{h}{2}$  and then it begins to grow non-linearly with time as stated below

$$I \approx I_{\text{vac}} + \frac{L^{D-1}}{4G_N \rho_H^{d-1+z}} \left( -c_1^z h^{z+1} + \frac{(zt)^{1+1/z}}{2z+2} \right). \quad (3.39)$$

To find the maximum value of the mutual information in this regime, the time behavior of  $S(\ell)$  plays a key role, it actually grows non-linearly till reaches to its equilibrium value which is given by (A.8), therefore one obtains an estimation for the time when the mutual information becomes maximum as follows

$$t_{\text{max}}^{(3)} \sim \frac{1}{z} \left( \frac{2z+2}{4(d-2)} c_1^z \right)^{\frac{z}{z+1}} \ell^z. \quad (3.40)$$

By making use of this maximum time one can obtain the maximum value of the mutual information as

$$I_{\text{max}}^{(3)} \approx I_{\text{vac}} + \frac{L^{D-1} c_1^z}{4G_N \rho_H^{d-1+z}} \left( \frac{\ell^{z+1}}{4(d-2)} - h^{z+1} \right). \quad (3.41)$$

Let us now study the other time intervals in more details.

#### 3.3.1 Non-linear decreasing $\frac{\ell}{2} \rho^{z-1} < t < \rho_H^z$

As already mentioned in this time interval the entanglement entropies of  $S(h)$  and  $S(\ell)$  are saturated to their equilibrium values which are given by (A.8), whereas  $S(2\ell + h)$  is still at the early times and should be approximated by equation (3.4)

$$S(2\ell + h) \approx \frac{L^{D-1}}{4G_N} \left[ \frac{1}{(d-2)\epsilon^{d-2}} - \frac{c_{d-1}}{(2\ell + h)^{d-2}} + \frac{m}{2(z+1)} (zt)^{1+\frac{1}{z}} \right]. \quad (3.42)$$

Plugging these results into equation (2.7), one finds

$$I \approx I_{\text{max}}^{(3)} + \frac{L^{D-1}}{4G_N \rho_H^{d-1+z}} \left( \frac{c_1^z}{2(d-2)} \ell^{z+1} - \frac{(zt)^{1+1/z}}{2(z+1)} \right). \quad (3.43)$$

Note that since  $t > t_{\text{max}}^{(3)}$  it is clear that  $I < I_{\text{max}}^{(3)}$ .

#### 3.3.2 Linear decreasing $\rho_H^z < t < \rho_H^{z-1}(\ell + \frac{h}{2})$

There is no change for the entanglement entropies  $S(h)$  and  $S(\ell)$ , however,  $S(2\ell + h)$  is locally equilibrated as already discussed in (3.11). Making use of the large entangling region limit for  $\rho_m$ , one can show that the mutual information reads as

$$I \approx I_{\text{max}}^{(3)} + \frac{L^{D-1}}{4G_N} \left( \frac{c_1^z}{4(d-2)\rho_H^{d-1+z}} \ell^{z+1} - v_E t \right). \quad (3.44)$$



We note that in this time interval the mutual information is positive as long as  $t < \rho_H^{z-1}(\ell + \frac{h}{2})$ .

### 3.3.3 Saturation

Again if one wait enough the system is going to be saturated. But in this time interval the entanglement entropies  $S(h)$  and  $S(\ell)$  have already been saturated here is no change in reading them and given by equation (A.8), though the entanglement entropy  $S(2\ell+h)$  must be approximated by equation (A.14). Therefore the mutual information can be recast as

$$I_{\text{sat}}^{(3)} \approx I_{\text{vac}} + \frac{L^{D-1}}{4G_N} \left( \left( \frac{c_1^z}{4(d-2)\rho_H^{d-1+z}} \right) (2\ell^{z+1} - h^{z+1}) - \frac{2\ell+h}{2\rho_H^{d-1}} + \frac{c_2}{\rho_H^{d-2}} - \frac{c_{d-2}}{(2\ell+h)^{d-2}} \right). \quad (3.45)$$

In the present regime, one can show that  $I_{\text{sat}}^{(3)} > I_{\text{vac}}$ . In order to estimate the saturation time, with the assumption of large entangling region and noting that the mutual information decreases linearly with time, one may equate equations (3.45) and (3.44) to find

$$t_s^{(3)} \sim \left( \ell + \frac{h}{2} \right) \rho_H^{z-1} - c_2 \rho_H^z. \quad (3.46)$$

To summarize the results of the third regime  $\frac{h}{2} \ll \frac{\ell}{2} < \rho_H < \ell + \frac{h}{2}$ , what we have obtained for the time evolution of the mutual information is as follows: at the very early time till  $t \sim \rho_H^{z-1} \frac{h}{2}$ , it has a constant value as it was in the vacuum state, then from  $t \sim \rho_H^{z-1} \frac{h}{2}$  to  $t_{\text{max}}^{(3)}$  it has non-linear growing behavior. After its maximum we have two declining phases non-linear which is followed by linear phase just before saturation. The mutual information saturates to a constant value at the saturation time  $t_s^{(3)}$ .

### 3.4 Fourth regime: $\frac{h}{2} \ll \frac{\ell}{2} < \ell + \frac{h}{2} \ll \rho_H$

The final case takes place when we are interested in a condition that all the entanglement entropies saturate to their equilibrium values before the system reaches a local equilibrium. The situation is very similar to the third regime, namely mutual information starts from its value in vacuum and remains fixed up to  $t \sim \rho_H^{z-1} \frac{h}{2}$  that it starts growing non-linearly with time which could be estimated as relation (3.39). This non-linear behavior lasts up to its maximum value which is given by

$$t_{\text{max}}^{(4)} \sim \frac{1}{z} \left( \frac{2z+2}{4(d-2)} c_1^z \right)^{\frac{z}{z+1}} \ell^z. \quad (3.47)$$

Similarly the maximum value reads as

$$I_{\text{max}}^{(4)} \approx I_{\text{vac}} + \frac{L^{D-1} c_1^z}{4G_N \rho_H^{d-1+z}} \left( \frac{\ell^{z+1}}{4(d-2)} - h^{z+1} \right). \quad (3.48)$$

After that the thermalization is followed by a nonlinear decreasing as

$$I \approx I_{\text{max}}^{(4)} + \frac{L^{D-1}}{4G_N \rho_H^{d-1+z}} \left( \frac{c_1^z}{2(d-2)} \ell^{z+1} - \frac{(zt)^{1+1/z}}{2(z+1)} \right). \quad (3.49)$$

Noting that since  $t > t_{\text{max}}^{(4)}$  again one gets  $I < I_{\text{max}}^{(4)}$ .

Finally after a long time the mutual information reaches its equilibrium value and the saturation takes place when all the entanglement entropies become that of a hyperscaling violating AdS black brane which is given by equation (A.8). Therefore from equation (2.7), the saturated mutual information is obtained as

$$I_{\text{sat}}^{(4)} \approx I_{\text{vac}} - \frac{L^{D-1}}{16G_N(d-2)} c_1^z \frac{(\ell+h)^{z+1}}{\rho_H^{d-1+z}}. \quad (3.50)$$

The saturation time can be estimated easily by assuming that the non-linear decreasing continues all the way to the saturation point then after equating eqs. (3.49) and (3.50) one can estimate the saturation time.

## 4 Conclusion

In this paper we studied the thermalization of the mutual information after a global quantum quench in a hyperscaling violating background. Within the language of the gauge/gravity duality a global quantum quench for a strongly coupled field theory with hyperscaling violation and an anisotropic scaling symmetry may be described by an AdS-Vaidya geometry with a hyperscaling violating factor at the gravity side and the quantum quench might be thought as an instant injection of matter in a small time interval. We used the covariant prescription for computing the holographic entanglement entropy and studied mutual information, holographically. This was done by extremizing a certain codimension-two hypersurface in the bulk whose metric is given by hyperscaling violating AdS-Vaidya geometry. In this study following ref. [1] we used two parallel strips with width  $\ell$  separated by distance  $h$  assuming that  $h \ll \ell$ . Actually thermalization depends on the evolution of the corresponding hypersurfaces of the entanglement entropies which is controlled by the region inside and around the horizon. Beside the radius of horizon the size of entangling regions can be used as a scale in probing the system. In other words the time behavior of the thermalization depends on relative size of the corresponding entangling regions and the radius of horizon. In fact depending that the size of entangling regions could be larger or smaller than the radius of horizon one may distinguish four main regimes. It goes as a general rule in which if the width of entangling region is smaller than the radius of horizon, the corresponding entanglement entropy grows non-linearly with time as  $t^{1+1/z}$  and saturates before the system reaches a local equilibrium. Noting that locally equilibrium occurs when the system has ceased production of thermodynamic entropy and the entanglement entropy can be given in terms of the thermal entropy. Another important case takes place when the width of the entangling region is larger than the radius of horizon, the corresponding entanglement entropy grows non-linearly with time before the system reaches a local equilibrium, with a linear growth after local equilibrium, it saturates.

It was argued that the early time behavior of the entanglement entropy depends on  $z$  while in the intermediate region its time behavior is linear [13]. We extended these statements for the thermalization of mutual information. We found that except the change in the time scale due to the different scaling of time, the very early time behavior is independent of  $z$  and is similar to what done in ref. [1], however, the non-linear behavior

is contorted by  $z$  and with  $z = 1$  one covers the previous results of quadratic behavior. On the other hand for large  $z$  the non-linear behavior turns to a linear behavior.

Actually understanding how quantum information spread in a strongly coupled system which is out of equilibrium is a question of much importance in many different areas of physics. In a 2-dimensional CFT, Calabrese and Cardy presented a simple physical picture for the linear growth behavior and saturation of the entanglement entropy [10]. In this model, entanglement entropy spreads via free propagation of EPR pairs of entangled *quasiparticles*, these ‘particles’ is assumed to be created by the injected energy density due to a global quench at  $t = 0$  and subsequently propagate freely with the speed of light. The gravity dual of this picture has been analyzed in [38]. Moreover, for a strongly coupled system with a gravity dual, the so-called entanglement *tsunami* proposal introduced in [12, 37, 39] could address the linear and quadratic behaviors of the entanglement entropy after a global quantum quench [40]. In this model the growth in entanglement entropy is described via a sharp wave-front carrying entanglement inward from the boundary of entangling region. Although the time scaling behavior of the entanglement entropy after a quench seems to be an open question, in the hyperscaling violating backgrounds, the linear growth behavior of the entanglement entropy is recovered, however, scaling like  $t^{1+\frac{1}{z}}$  indicates that at early time after quench the entanglement entropy is sensitive to the initial state.

In the literature of quantum information the subjects say as  $n$ -partite information or multi-partite entanglement become important [41]. When we are interested in measurement of the amount of information or correlations (both classical and quantum) between  $n$  disjoint regions  $A_i$ ,  $i = 1, \dots, n$ , the  $n$ -partite information may provide a good model for studying this process. It is shown that both  $n$ -partite information or multi-partite entanglement can be written in terms of the mutual information. The results of present work can be extended to those subjects to study their non-equilibrium behaviors.

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## A A brief review of entanglement entropy in hyperscaling violating backgrounds

As it is known in a given system with Hilbert space  $H$ , the states are characterized by the density matrix  $\rho$ . Now suppose this system can be divided by two subsystems  $A$  (which mostly supposed as a spatial region) and  $\bar{A}$  which stands for the complement of  $A$ , so that one can write  $H = H_A \otimes H_{\bar{A}}$ . The reduced density matrix of  $A$  can be obtained by tracing over the degrees of freedom of  $\bar{A}$ , then the entanglement entropy of  $A$  is defined by the von Neumann entropy of the corresponding reduced density matrix:  $S_A = -\text{Tr}(\rho_A \log \rho_A)$ .

Entanglement entropy has a UV divergent term where in spatial dimensions bigger than one, the divergent term obeys the area law, namely it is proportional to the area of the entangling region [42], while for two dimensional spacetimes the divergent term is logarithmic (see for example [43, 44] for two dimensional CFT). In the context of the quantum field theory the replica trick has been mostly used to obtain the entanglement entropy which is based on computing the Renyi entropies (for details see [45]). However, for theories which have gravity dual, the Ryu-Takayanagi conjecture has been used to calculate the entanglement entropy. Noting that in the case of the time-dependent backgrounds one should use the covariant proposal of the entanglement entropy [46].

### A.1 Static hyperscaling violating backgrounds

According to the holographic description for a given entangling region, the entanglement entropy is given by

$$S = \frac{\mathcal{A}}{4G_N^{D+2}}, \quad (\text{A.1})$$

where  $G_N$  is the Newton constant and  $\mathcal{A}$  is the  $D$ -dimensional minimal surface in the bulk whose boundary coincides with the boundary of the entangling region. Let us choose a strip at a fixed time on the boundary as an entangling region as indicated below

$$-\frac{\ell}{2} \leq x_1 \equiv x \leq \frac{\ell}{2}, \quad 0 \leq x_a \leq L, \quad a = 2, \dots, D, \quad (\text{A.2})$$

where  $(t, \vec{x})$  are the space-time coordinates. Now the aim is to find the surface in the bulk with the boundary of the above strip and then minimize it. The corresponding metric is given by (2.2) and the profile of the hypersurface in the bulk may be parameterized by  $x(r)$ , after setting  $r = \rho^{-1}$ , the induced metric on this hypersurface is given by

$$ds_{\text{ind}}^2 = \rho^{2\frac{D}{D-2}} \left[ \left( \frac{1}{f(\rho)} + x'^2 \right) d\rho^2 + \sum_{a=2}^D dx_a^2 \right] \quad (\text{A.3})$$

where here the prime stands for the derivative with respect to  $\rho$  and  $f(\rho)$  is as follows

$$f(\rho) = 1 - m\rho^{d-1+z} = 1 - \left( \frac{\rho}{\rho_H} \right)^{d-1+z}. \quad (\text{A.4})$$

Thus the area reads as

$$\mathcal{A} = \frac{L^{D-1}}{2} \int d\rho \frac{\sqrt{f^{-1} + x'^2}}{\rho^{d-1}}, \quad (\text{A.5})$$

where one should minimize the area and hence one obtains

$$\frac{\ell}{2} = \int_0^{\rho_t} d\rho \frac{\left( \frac{\rho}{\rho_t} \right)^{d-1}}{\sqrt{f(\rho) \left( 1 - \left( \frac{\rho}{\rho_t} \right)^{2(d-1)} \right)}}, \quad (\text{A.6})$$

note that  $\rho_t$  is the turning point of the extremal hypersurface in the bulk. Therefore, if one introduces  $\epsilon$  to be a UV cutoff of theory the entropy becomes

$$S = \frac{L^{D-1}}{4G_N} \int_{\epsilon}^{\rho_t} d\rho \frac{1}{\rho^{d-1} \sqrt{f(\rho) \left( 1 - \left( \frac{\rho}{\rho_t} \right)^{2(d-1)} \right)}}. \quad (\text{A.7})$$

$z$ and $d$	$z = 1, d = 3$	$z = 1, d = 4$	$z = 1, d = 5$	$z = 2, d = 3$	$z = 2, d = 4$
$c_2$	0.88	0.33	0.14	1	0.44

**Table 1.**  $c_2$  in  $S_{BH}$  with certain  $z$  and  $d$ .

In general there is no explicit analytic expression for the entanglement entropy, however, in some certain limits say as  $m\ell^{d-1+z} \ll 1$ , one can expand the area (A.5) around the  $f = 1$  which leads to the following expression for the change of the entanglement entropy

$$\Delta S = S_{BH} - S_{\text{vac}} = \frac{L^{D-1}}{16G_N(d-2)} c_1^z \frac{\ell^{z+1}}{\rho_H^{d-1+z}}, \quad (\text{A.8})$$

where

$$c_1 = \frac{\Gamma(\frac{1}{2(d-2)})}{2\sqrt{\pi}\Gamma(\frac{d}{2(d-1)})}. \quad (\text{A.9})$$

In fact this is the change in the entanglement entropy due to the change of the geometry between black hole solution  $f \neq 1$  and vacuum solution  $f = 1$ . For a strip the entanglement entropy for vacuum was found in [11]

$$S_{\text{vac}} = \begin{cases} \frac{L^{D-1}}{4G_N} \left( \frac{1}{(d-2)\epsilon^{d-2}} - \frac{c_{d-1}}{\ell^{d-2}} \right) & \text{for } d > 2, \\ \frac{L^{D-1}}{4G_N} \ln \frac{\ell}{\epsilon}, & \text{for } d = 2, \end{cases} \quad (\text{A.10})$$

where

$$c_{d-1} = \frac{2^{d-2}}{d-2} \left( \frac{\sqrt{\pi}\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})} \right)^{d-1}. \quad (\text{A.11})$$

On the other hand for large entangling regions  $m\ell^{d-1+z} \gg 1$ , the main contribution comes from the limit where the minimal surface is extended to the horizon, namely  $\rho_t \sim \rho_H$ . In this limit after making use of the expansion of the equations (A.6) and (A.7) one finds

$$S_{BH} \sim \frac{L^{D-1}}{4G_N} \frac{1}{\rho_H^{d-1}} \frac{\ell}{2} + \frac{L^{D-1}}{4G_N} \frac{1}{\rho_H^{d-2}} \int_{\frac{\epsilon}{\rho_H}}^1 \frac{\sqrt{1 - \xi^{2(d-1)}} d\xi}{\sqrt{1 - \xi^{d-1+z}} \xi^{d-1}}, \quad (\text{A.12})$$

where  $\xi$  is a dimensionless parameter defined by  $\xi \equiv \frac{\rho}{\rho_H}$ . The above integral can not be solved, however by extracting its divergence, it can be written as

$$\int_{\frac{\epsilon}{\rho_H}}^1 \frac{d\xi}{\xi^{d-1} \sqrt{(1 - \xi^{d-1+z})(1 - \xi^{2(d-1)})}} = \frac{1}{(d-2)\epsilon^{d-2}} - c_2, \quad (\text{A.13})$$

where for a fixed  $z$  and  $d$  it can be computed, as shown in below

Therefore the entanglement entropy reads as

$$S_{BH} \approx \frac{L^{D-1}}{4G_N} \left( \frac{1}{(d-2)\epsilon^{d-2}} + \frac{\ell}{2\rho_H^{d-1}} - \frac{c_2}{\rho_H^{d-2}} \right). \quad (\text{A.14})$$

## A.2 Time-dependent hyperscaling violating backgrounds

Entanglement entropy could potentially provide useful quantity of time scaling in a system undergoing a rapid change (quantum quench) which this change brings the system out of equilibrium. In our case in hand, as mentioned previously, at the gravity side the Vaidya metric can well describe an infalling shell of a massless and pressureless matter in a hyperscaling violating geometry. The corresponding metric is

$$ds_{D+2}^2 = \rho^{2\frac{\theta-D}{D}} \left( -\rho^{2-2z} f(\rho, v) dv^2 - 2\rho^{1-z} d\rho dv + \sum_{i=1}^D dx_i^2 \right), \quad (\text{A.15})$$

in which we have employed the Eddington-Finkelstein-like coordinate in (2.2) as defined by

$$dv = dt + \frac{dr}{f(r)r^{z+1}}, \quad (\text{A.16})$$

which followed by setting  $r = \rho^{-1}$  (for mathematical details see [13] and ref.s therein). With the effective dimension one has  $f(\rho, v) = 1 - m(v)\rho^{d-1+z}$ .

For a system after a global quantum quench, the entanglement entropy could provide a proper scaling during the process of the thermalization. In the process of evolution two phases become crucial: a time growing phase and a saturation phase where the entanglement entropy saturates to its equilibrium value. The radius of the horizon  $\rho_H$ , and the size of the entangling region can be used as scales to address these two phases. For a strip as an entangling region, the size is given by its width  $\ell$ . We use a strip with width  $\ell$  defined in (A.2) as an entangling region. Following the holographic description, in computing the entanglement entropy one should use the covariant proposal of finding the extremum area of a codimension-two hypersurface whose boundary coincides with the above mentioned strip [46]. At this stage one only needs to find the extremal surface of the codimension-two hypersurface in the bulk. To do so, we use the  $v(x)$  and  $\rho(x)$  to parameterize the corresponding codimension-two hypersurface in the bulk. Using the metric (A.15) which is the bulk metric, one finds the induced metric on the hypersurface as

$$ds_{\text{ind}}^2 = \rho^{2\frac{1-d}{D}} \left[ \left( 1 - \rho^{2-2z} f(\rho, v) v'^2 - 2\rho^{1-z} v' \rho' \right) dx^2 + \sum_{a=2}^D dx_a^2 \right], \quad (\text{A.17})$$

noting that prime represents derivative with respect to  $x$ . With this induced metric, the hypersurface area becomes

$$\mathcal{A} = \frac{L^{D-1}}{2} \int_{-\ell/2}^{\ell/2} dx \frac{\sqrt{1 - 2\rho^{1-z} v' \rho' - \rho^{2-2z} v'^2 f}}{\rho^{d-1}}, \quad (\text{A.18})$$

which should be extremized. However, the area expression (A.18) can be treated as an action of a one-dimensional quantum mechanical system where the  $v(x)$  and  $\rho(x)$  are its dynamical fields. The supposed action is independent of  $x$  and hence the corresponding Hamiltonian becomes a constant of motion and results in a conservation law

$$H^{-1} \equiv -\rho^{d-1} \sqrt{1 - 2\rho^{1-z} v' \rho' - \rho^{2-2z} v'^2 f} = \text{const.} \quad (\text{A.19})$$

Therefore, the equations of motion read as

$$\partial_x P_v = \frac{P_\rho^2}{2} \frac{\partial f}{\partial v}, \quad \partial_x P_\rho = \frac{P_\rho^2}{2} \frac{\partial f}{\partial \rho} + \frac{d-1}{\rho^{2d-1}} H^{-2} + \frac{1-z}{\rho^{2-z}} P_\rho P_v, \quad (\text{A.20})$$

where  $P$ 's stand for the the momenta conjugate defined by (up to a factor of  $H$ )

$$P_\rho = \rho^{1-z} v', \quad P_v = \rho^{1-z} (\rho' + \rho^{1-z} v' f).$$

If  $(\rho_t, v_t)$  is the turning point of the extremal hypersurface in the bulk one can use the following boundary conditions

$$\rho(0) = \rho_t, \quad v(0) = v_t, \quad v\left(\frac{\ell}{2}\right) = t, \quad \rho\left(\frac{\ell}{2}\right) = \rho'(0) = v'(0) = 0. \quad (\text{A.21})$$

In general there is no analytic solution for eqs. (A.20) but, in a special form of  $m$  say as  $m(v) = m \theta(v)$  one can solve the equations, where  $\theta(v)$  is the step function and therefore for  $v < 0$  the geometry (1.4) is an AdS metric while for  $v > 0$  it is an AdS-Schwarzschild black hole whose horizon is located at  $\rho_H = m^{-1/(D+1)}$ . Quench or a sudden change in a strongly coupled field theory could be modeled by the  $\theta$ -function in the gravity side, let suppose it as follows

$$f(\rho, v) = 1 - \theta(v) \left( \frac{\rho}{\rho_H} \right)^{d-1+z}. \quad (\text{A.22})$$

Now the aim is to solve eqs. (A.20), however because of the step function, three region can be distinguished as stated below:

- Region I:  $v < 0$

In this region theta-function is zero so that  $f(\rho, v) = 1$  this means that the system is in its vacuum state. The corresponding gravity dual is given by

$$ds^2 = \rho^{\frac{2(1-d)}{D}} \left( -\rho^{2-2z} dv^2 - 2\rho^{1-z} d\rho dv + d\vec{x}^2 \right). \quad (\text{A.23})$$

Having noted that  $\frac{\partial f(\rho, v)}{\partial v} = 0$ , from the first equation of (A.20) and the boundary condition (A.21), the momentum conjugate of  $v$  is found to be a constant equal to zero

$$P_{(1)v} = \rho^{1-z} (\rho' + \rho^{1-z} v') = 0. \quad (\text{A.24})$$

Note that we use the index (1) referring the value of quantities in the first case namely  $v < 0$  region. From the above equation and after making use of the relation (A.19), one obtains the following profile of the extremal surface

$$v(\rho) = v_t + \frac{1}{z} (\rho_t^z - \rho^z), \quad x(\rho) = \int_\rho^{\rho_t} \frac{\xi^{d-1} d\xi}{\sqrt{\rho_t^{2(d-1)} - \xi^{2(d-1)}}}. \quad (\text{A.25})$$

At the null shell one has  $v = 0$ , consequently from the above equation one can obtain the point where hypersurface intersects null shell as

$$\rho_c^z = \rho_t^z + z v_t. \quad (\text{A.26})$$

On the other hand one can also obtain

$$\rho'_{(1)} = -\rho_c^{1-z} v'_{(1)} = -\sqrt{\left(\frac{\rho_t}{\rho_c}\right)^{2(d-1)} - 1}. \quad (\text{A.27})$$

Note that  $(\rho_t, v_t)$  refers to the turning point of the extremal hypersurface in the bulk whereas  $(\rho_c, v_c)$  is the crossing point of the hypersurface and null shell.

- Region II:  $v > 0$

In this region one has  $f(\rho) = 1 - \left(\frac{\rho}{\rho_H}\right)^{d-1+z} \equiv 1 - g(\rho)$  and hence the corresponding geometry becomes a static black brane which is given by

$$ds^2 = \rho^{\frac{2(1-d)}{D}} \left( -\rho^{2-2z} f(\rho) dv^2 - 2\rho^{1-z} d\rho dv + d\vec{x}^2 \right). \quad (\text{A.28})$$

Once again one has  $\frac{\partial f}{\partial v} = 0$ , so that the corresponding momentum conjugate of  $v$  becomes a constant

$$P_{(2)v} = \rho^{1-z} (\rho' + \rho^{1-z} v' f(\rho)) = \text{const.} \quad (\text{A.29})$$

Similarly, from the conservation law (A.19) it can be shown

$$\rho'^2 = \frac{P_{(2)v}^2}{\rho^{2-2z}} + \left( \left( \frac{\rho_t}{\rho} \right)^{2(d-1)} - 1 \right) f(\rho) \equiv V_{\text{eff}}(\rho), \quad (\text{A.30})$$

where  $V_{\text{eff}}(\rho)$  might be considered as an effective potential describing a one-dimensional dynamical system with a dynamical variable  $\rho$ , this assumption helps us in describing the behavior of the entanglement entropy. Using this effective potential one obtains

$$\frac{dv}{d\rho} = -\frac{1}{\rho^{2(1-z)} f(\rho)} \left( \rho^{1-z} + \frac{P_{(2)v}}{\sqrt{V_{\text{eff}}(\rho)}} \right). \quad (\text{A.31})$$

- Null Shell  $v = 0$

At the null shell, one should consider the matching of the results of two previous regions, noting that  $\rho$  and  $v$  which are the space-time coordinates are indeed continues across the null shell. It is worth to mention that since matter has been injected along  $v$  namely the null direction, the corresponding momentum conjugate jumps once one moves from the initial phase ( $v > 0$ ) to the final phase ( $v < 0$ ). However, the momentum conjugate of  $\rho$  remains continuous which means  $v'_{(1)} = v'_{(2)}$ . Taking the integration of the equations of motion across the null shell results in

$$\rho'_{(2)} = \left( 1 - \frac{1}{2} g(\rho_c) \right) \rho'_{(1)}, \quad \mathcal{L}_{(1)} = \mathcal{L}_{(2)}. \quad (\text{A.32})$$

Thus the momentum conjugate of  $v$  becomes:

$$P_{(2)v} = \frac{1}{2} \rho_c^{1-z} g(\rho_c) \rho'_{(1)} = -\frac{1}{2} \rho_c^{1-z} g(\rho_c) \sqrt{\left(\frac{\rho_t}{\rho_c}\right)^{2(d-1)} - 1}. \quad (\text{A.33})$$



To compute the extremal hypersurface in the bulk, it is important to mention that the extremal hypersurface could be in fact extended in both  $v < 0$  and  $v > 0$  regions. Accordingly the width  $\ell$  and the boundary time are given by

$$\frac{\ell}{2} = \int_{\rho_c}^{\rho_t} \frac{d\rho \rho^{d-1}}{\sqrt{\rho_t^{2d-2} - \rho^{2d-2}}} + \int_0^{\rho_c} \frac{d\rho}{\sqrt{V_{\text{eff}}}}, \quad t = \int_0^{\rho_c} \frac{d\rho \rho^{z-1}}{f(\rho)} \left( 1 + \frac{E \rho^{z-1}}{\sqrt{V_{\text{eff}}(\rho)}} \right), \quad (\text{A.34})$$

where  $E = P_{(2)v} \rho_t^{z-1}$ . Therefore the area of the hypersurface in the bulk is found to be

$$\mathcal{A}_{d-1} = \frac{L^{D-1}}{\rho_t^{d-2}} \left( \int_{\frac{\rho_c}{\rho_t}}^1 \frac{d\xi}{\xi^{d-1} \sqrt{1 - \xi^{2(d-1)}}} + \int_0^{\frac{\rho_c}{\rho_t}} \frac{d\xi}{\xi^{2(d-1)} \sqrt{R(\xi)}} \right), \quad (\text{A.35})$$

where

$$R(\xi) \equiv V_{\text{eff}}(\rho_t \xi) = E^2 \xi^{2(z-1)} + \left( \frac{1}{\xi^{2(d-1)}} - 1 \right) f(\rho_t \xi), \quad (\text{A.36})$$

in which  $f(\rho_t \xi) = 1 - (\frac{\rho_t}{\rho_H})^{d-1+z} \xi^{d-1+z}$ . Clearly the area (A.35) for large volume is divergent (UV effect) and a proper UV cutoff is actually needed. In fact what we are interested in is the change of the area when the system evolves from its vacuum state to an excited state namely, corresponding area of the extremal hypersurface in the vacuum solution is given by

$$\mathcal{A}_{d-1}^{\text{vac}} = \frac{L^{D-1}}{\rho_t^{d-2}} \int_0^1 \frac{d\xi}{\xi^{d-1} \sqrt{1 - \xi^{2d-2}}}, \quad (\text{A.37})$$

which it could play the role of a regulator. One can use the expression for  $t$ ,  $\ell$  and  $\mathcal{A}$  to study the scaling behavior of the entanglement entropy during the process of the thermalization after a global quantum quench.

### A.3 Saturation: late time equilibrium

After a long time from a change due to the quench, system reaches to the thermal equilibrium and hence the entanglement entropy saturates to its equilibrium value. In fact the system is locally equilibrated for  $t \geq \rho_H$  and the saturation takes place when the extremal hypersurface is entirely outside the horizon namely  $\rho_t < \rho_H$ . In the saturation time as long as we are dealing with the large entangling region, the main contribution of the hypersurface in the bulk comes from the geometry around the horizon, namely one has  $\rho_c \simeq \rho_t \simeq \rho_H$ . Now the aim is to compute the area and the entanglement entropy in this limit. By taking  $\rho_c = \rho_t(1 - \delta)$  for  $\delta \ll 1$  and also noting that  $P_{(2)v} = 0$  and also  $\rho_t \simeq \rho_H$ , one can expand (A.34) and (A.35) in this limit as

$$\frac{\ell}{2} \approx \rho_H \int_0^{1-\delta} \frac{d\xi}{\sqrt{V_{\text{eff}}(\rho_t \xi)}}, \quad \mathcal{A} \approx \frac{L^{D-1}}{\rho_H^{d-2}} \int_{\frac{\epsilon}{\rho_H}}^{1-\delta} \frac{d\xi}{\xi^{2(d-1)} \sqrt{V_{\text{eff}}(\rho_t \xi)}}. \quad (\text{A.38})$$

Noting that the main contribution to the  $\ell$  and  $\mathcal{A}$  come from the  $\xi = 1$ , then one can write

$$\mathcal{A} \approx \frac{L^{D-1}}{\rho_H^{d-2}} \left( \int_0^{1-\delta} \frac{d\xi}{\sqrt{V_{\text{eff}}(\rho_t \xi)}} + \int_{\frac{\epsilon}{\rho_H}}^1 \frac{(1 - \xi^{2(d-1)}) d\xi}{\xi^{2(d-1)} \sqrt{V_{\text{eff}}(\rho_t \xi)}} \right). \quad (\text{A.39})$$

The second term diverges at the UV limit and it can be regularized as (A.13) then one can write

$$\mathcal{A} \approx \frac{L^{D-1}}{\rho_H^{d-2}} \left( \frac{\ell}{2\rho_H} + \frac{1}{(d-2)\epsilon^{d-2}} - c_2 \right), \quad (\text{A.40})$$

thus the saturated entanglement entropy can be written accordingly.

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